REVIEW EXERCISES AND PROBLEMS FOR CHAPTER TWO

Exercises

In Exercises 1–6, find the average velocity for the position function $s(t)$, in mm, over the interval $1 \le t \le 3$, where t is in seconds.

1. $s(t) = 12t - t^2$

2. $s(t) = \ln(t)$

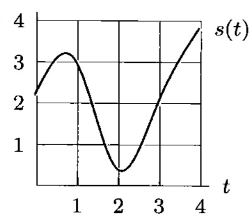
3.

t	0	1	2	3
$s(t)$	7	3	7	11

4.

t	0	1	2	3
$s(t)$	8	4	2	4

5.



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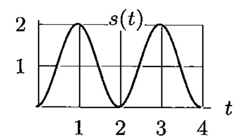

Sketch the graphs of the derivatives of the functions shown in Exercises 7–12. Be sure your sketches are consistent with the important features of the graphs of the original functions.

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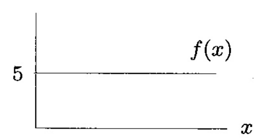

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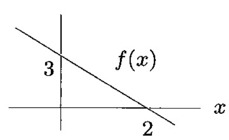

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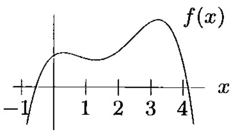

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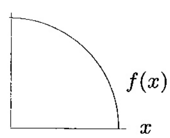

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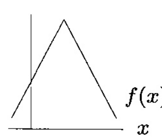

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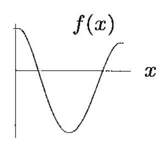

In Exercises 13–14, graph the second derivative of each of the given functions.

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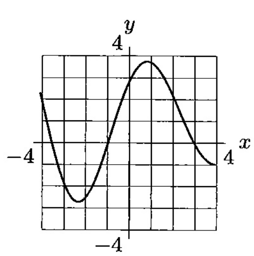

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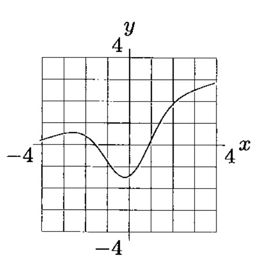

Find a formula for the derivatives of the functions in Exercises 15–16 algebraically.

15. $f(x) = 5x^2 + x$

16. $n(x) = (1/x) + 1$

17. Find the derivative of $f(x) = x^2 + 1$ at $x = 3$ algebraically. Find the equation of the tangent line to f at $x = 3$.

18. **(a)** Between which pair of consecutive points in Figure 2.62 is the average rate of change of k

(a) Greatest? **(b)** Closest to zero?

(b) Between which two pairs of consecutive points are the average rates of change of k closest?

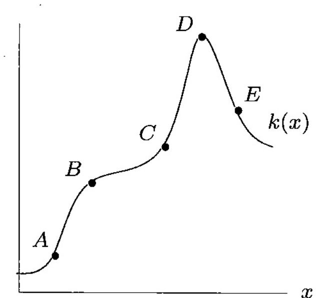

Figure 2.62

Use algebra to evaluate the limits in Exercises 19–23. Assume $a > 0$.

19. $\lim_{h \to 0} \dfrac{(a+h)^2 - a^2}{h}$

20. $\lim_{h \to 0} \dfrac{1/(a+h) - 1/a}{h}$

21. $\lim_{h \to 0} \dfrac{1/(a+h)^2 - 1/a^2}{h}$

22. $\lim_{h \to 0} \dfrac{\sqrt{a+h} - \sqrt{a}}{h}$ [Hint: Multiply by $\sqrt{a+h} + \sqrt{a}$ in numerator and denominator.]

23. $\lim_{h \to 0} \dfrac{1/\sqrt{a+h} - 1/\sqrt{a}}{h}$

Problems

24. Sketch the graph of a function whose first and second derivatives are everywhere positive.

25. Figure 2.63 gives the position, $y = s(t)$, of a particle at time t. Arrange the following numbers from smallest to largest:

- The instantaneous velocity at A.
- The instantaneous velocity at B.
- The instantaneous velocity at C.
- The average velocity between A and B.
- The number 0.
- The number 1.

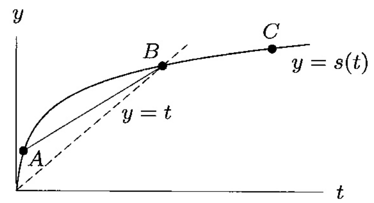

Figure 2.63

26. The table[12] gives the number of passenger cars, $C = f(t)$, in millions, in the US in the year t.

(a) During the period 2002–2006, when is $f'(t)$ positive? Negative?
(b) Estimate $f'(2003)$. Using units, interpret your answer in terms of passenger cars.

t (year)	2002	2003	2004	2005	2006
C (cars, in millions)	135.9	135.7	136.4	136.6	135.4

27. Let $f(t)$ be the depth, in centimeters, of water in a tank at time t, in minutes.

(a) What does the sign of $f'(t)$ tell us?
(b) Explain the meaning of $f'(30) = 20$. Include units.
(c) Using the information in part (b), at time $t = 30$ minutes, find the rate of change of depth, in meters, with respect to time in hours. Give units.

28. The revenue, in thousands of dollars, earned by a gas station when the price of gas is \$$p$ per gallon is $R(p)$.

(a) What are the units of $R'(3)$? Interpret this quantity.
(b) What are the units of $(R^{-1})'(5)$? Interpret this quantity.

29. **(a)** Give an example of a function with $\lim_{x\to 2} f(x) = \infty$.
(b) Give an example of a function with $\lim_{x\to 2} f(x) = -\infty$.

[12]www.bts.gov, accessed May 27, 2008.

30. Suppose $f(2) = 3$ and $f'(2) = 1$. Find $f(-2)$ and $f'(-2)$, assuming that $f(x)$ is

(a) Even **(b)** Odd.

31. Do the values for the function $y = k(x)$ in the table suggest that the graph of $k(x)$ is concave up or concave down for $1 \le x \le 3.3$? Write a sentence in support of your conclusion.

x	1.0	1.2	1.5	1.9	2.5	3.3
$k(x)$	4.0	3.8	3.6	3.4	3.2	3.0

32. Suppose that $f(x)$ is a function with $f(20) = 345$ and $f'(20) = 6$. Estimate $f(22)$.

33. Students were asked to evaluate $f'(4)$ from the following table which shows values of the function f:

x	1	2	3	4	5	6
$f(x)$	4.2	4.1	4.2	4.5	5.0	5.7

- Student A estimated the derivative as $f'(4) \approx \dfrac{f(5)-f(4)}{5-4} = 0.5$.
- Student B estimated the derivative as $f'(4) \approx \dfrac{f(4)-f(3)}{4-3} = 0.3$.
- Student C suggested that they should split the difference and estimate the average of these two results, that is, $f'(4) \approx \frac{1}{2}(0.5 + 0.3) = 0.4$.

(a) Sketch the graph of f, and indicate how the three estimates are represented on the graph.
(b) Explain which answer is likely to be best.
(c) Use Student C's method to find an algebraic formula to approximate $f'(x)$ using increments of size h.

34. Use Figure 2.64 to fill in the blanks in the following statements about the function f at point A.

(a) $f(\_\_) = \_\_$ **(b)** $f'(\_\_) = \_\_$

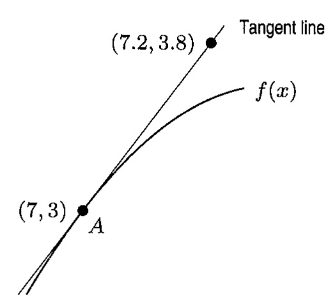

Figure 2.64

35. Use Figure 2.65. At point A, we are told that $x = 1$. In addition, $f(1) = 3$, $f'(1) = 2$, and $h = 0.1$. What are the values of $x_1, x_2, x_3, y_1, y_2, y_3$?

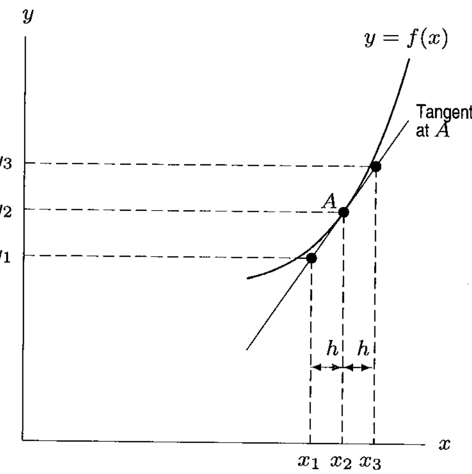

Figure 2.65

36. Given all of the following information about a function f, sketch its graph.

- $f(x) = 0$ at $x = -5, x = 0$, and $x = 5$
- $\lim_{x\to-\infty} f(x) = \infty$
- $\lim_{x\to\infty} f(x) = -3$
- $f'(x) = 0$ at $x = -3, x = 2.5$, and $x = 7$

37. A yam has just been taken out of the oven and is cooling off before being eaten. The temperature, T, of the yam (measured in degrees Fahrenheit) is a function of how long it has been out of the oven, t (measured in minutes). Thus, we have $T = f(t)$.

(a) Is $f'(t)$ positive or negative? Why?
(b) What are the units for $f'(t)$?

38. An economist is interested in how the price of a certain commodity affects its sales. Suppose that at a price of $\$p$, a quantity q of the commodity is sold. If $q = f(p)$, explain in economic terms the meaning of the statements $f(10) = 240{,}000$ and $f'(10) = -29{,}000$.

39. At time, t, in years, the US population is growing at 0.8% per year times its size, $P(t)$, at that moment. Using the derivative, write an equation representing this statement.

40. **(a)** Using the table, estimate $f'(0.6)$ and $f'(0.5)$.
(b) Estimate $f''(0.6)$.
(c) Where do you think the maximum and minimum values of f occur in the interval $0 \le x \le 1$?

x	0	0.2	0.4	0.6	0.8	1.0
$f(x)$	3.7	3.5	3.5	3.9	4.0	3.9

41. Let $g(x) = \sqrt{x}$ and $f(x) = kx^2$, where k is a constant.

(a) Find the slope of the tangent line to the graph of g at the point $(4, 2)$.
(b) Find the equation of this tangent line.
(c) If the graph of f contains the point $(4, 2)$, find k.
(d) Where does the graph of f intersect the tangent line found in part (b)?

42. A circle with center at the origin and radius of length $\sqrt{19}$ has equation $x^2 + y^2 = 19$. Graph the circle.

(a) Just from looking at the graph, what can you say about the slope of the line tangent to the circle at the point $(0, \sqrt{19})$? What about the slope of the tangent at $(\sqrt{19}, 0)$?
(b) Estimate the slope of the tangent to the circle at the point $(2, -\sqrt{15})$ by graphing the tangent carefully at that point.
(c) Use the result of part (b) and the symmetry of the circle to find slopes of the tangents drawn to the circle at $(-2, \sqrt{15})$, $(-2, -\sqrt{15})$, and $(2, \sqrt{15})$.

43. Each of the graphs in Figure 2.66 shows the position of a particle moving along the x-axis as a function of time, $0 \le t \le 5$. The vertical scales of the graphs are the same. During this time interval, which particle has

(a) Constant velocity?
(b) The greatest initial velocity?
(c) The greatest average velocity?
(d) Zero average velocity?
(e) Zero acceleration?
(f) Positive acceleration throughout?

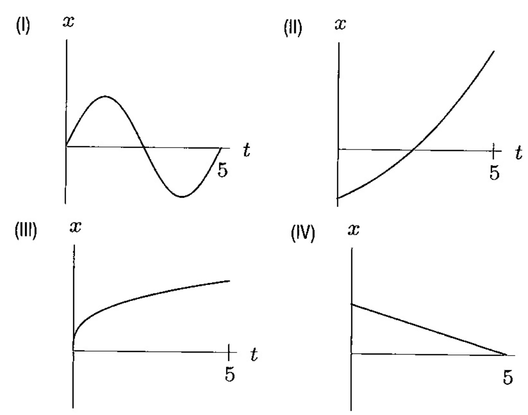

Figure 2.66

44. The population of a herd of deer is modeled by

$$P(t) = 4000 + 400\sin\left(\frac{\pi}{6}t\right) + 180\sin\left(\frac{\pi}{3}t\right)$$

where t is measured in months from the first of April.

(a) Use a calculator or computer to sketch a graph showing how this population varies with time.

Use the graph to answer the following questions.

(b) When is the herd largest? How many deer are in it at that time?

(c) When is the herd smallest? How many deer are in it then?

(d) When is the herd growing the fastest? When is it shrinking the fastest?

(e) How fast is the herd growing on April 1?

45. The number of hours, H, of daylight in Madrid is a function of t, the number of days since the start of the year. Figure 2.67 shows a one-month portion of the graph of H.

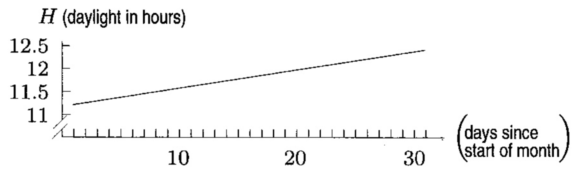

Figure 2.67

(a) Comment on the shape of the graph. Why does it look like a straight line?

(b) What month does this graph show? How do you know?

(c) What is the approximate slope of this line? What does the slope represent in practical terms?

46. Suppose you put a yam in a hot oven, maintained at a constant temperature of 200°C. As the yam picks up heat from the oven, its temperature rises.[13]

(a) Draw a possible graph of the temperature T of the yam against time t (minutes) since it is put into the oven. Explain any interesting features of the graph, and in particular explain its concavity.

(b) Suppose that, at $t = 30$, the temperature T of the yam is $120°$ and increasing at the (instantaneous) rate of $2°$/min. Using this information, plus what you know about the shape of the T graph, estimate the temperature at time $t = 40$.

(c) Suppose in addition you are told that at $t = 60$, the temperature of the yam is $165°$. Can you improve your estimate of the temperature at $t = 40$?

(d) Assuming all the data given so far, estimate the time at which the temperature of the yam is $150°$.

47. You are given the following values for the error function, $\text{erf}(x)$.

$$\text{erf}(0) = 0 \quad \text{erf}(1) = 0.84270079$$
$$\text{erf}(0.1) = 0.11246292 \quad \text{erf}(0.01) = 0.01128342.$$

(a) Use all this information to determine your best estimate for $\text{erf}'(0)$. (Give only those digits of which you feel reasonably certain.)

(b) Suppose you find that $\text{erf}(0.001) = 0.00112838$. How does this extra information change your answer to part (a)?

48. **(a)** Use your calculator to approximate the derivative of the hyperbolic sine function (written $\sinh x$) at the points 0, 0.3, 0.7, and 1.

(b) Can you find a relation between the values of this derivative and the values of the hyperbolic cosine (written $\cosh x$)?

CAS Challenge Problems

49. Use a computer algebra system to find the derivative of $f(x) = \sin^2 x + \cos^2 x$ and simplify your answer. Explain your result.

50. **(a)** Use a computer algebra system to find the derivative of $f(x) = 2\sin x \cos x$.

(b) Simplify $f(x)$ and $f'(x)$ using double angle formulas. Write down the derivative formula that you get after doing this simplification.

51. **(a)** Use a computer algebra system to find the second derivative of $g(x) = e^{-ax^2}$ with respect to x.

(b) Graph $g(x)$ and $g''(x)$ on the same axes for $a = 1, 2, 3$ and describe the relation between the two graphs.

(c) Explain your answer to part (b) in terms of concavity.

52. **(a)** Use a computer algebra system to find the derivative of $f(x) = \ln(x)$, $g(x) = \ln(2x)$, and $h(x) = \ln(3x)$. What is the relationship between the answers?

(b) Use the properties of logarithms to explain what you see in part (a).

53. **(a)** Use a computer algebra system to find the derivative of $(x^2+1)^2$, $(x^2+1)^3$, and $(x^2+1)^4$.

(b) Conjecture a formula for the derivative of $(x^2+1)^n$ that works for any integer n. Check your formula using the computer algebra system.

54. **(a)** Use a computer algebra system to find the derivatives of $\sin x$, $\cos x$ and $\sin x \cos x$.

(b) Is the derivative of a product of two functions always equal to the product of their derivatives?

[13]From Peter D. Taylor, *Calculus: The Analysis of Functions* (Toronto: Wall & Emerson, Inc., 1992).

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–22 true or false? Give an explanation for your answer.

1. If a car is going 50 miles per hour at 2 pm and 60 miles per hour at 3 pm then it travels between 50 and 60 miles during the hour between 2 pm and 3 pm.
2. If a car travels 80 miles between 2 and 4 pm, then its velocity is close to 40 mph at 2 pm.
3. If the time interval is short enough, then the average velocity of a car over the time interval and the instantaneous velocity at a time in the interval can be expected to be close.
4. If an object moves with the same average velocity over every time interval, then its average velocity equals its instantaneous velocity at any time.
5. The formula Distance traveled = Average velocity × Time is valid for every moving object for every time interval.
6. By definition, the instantaneous velocity of an object equals a difference quotient.
7. If $f(x)$ is concave up, then $f'(a) < (f(b) - f(a))/(b - a)$ for $a < b$.
8. You cannot be sure of the exact value of a derivative of a function at a point using only the information in a table of values of the function. The best you can do is find an approximation.
9. If $f'(x)$ is increasing, then $f(x)$ is also increasing.
10. If $f(a) \neq g(a)$, then $f'(a) \neq g'(a)$.
11. The derivative of a linear function is constant.
12. If $g(x)$ is a vertical shift of $f(x)$, then $f'(x) = g'(x)$.
13. If $f(x)$ is defined for all x but $f'(0)$ is not defined, then the graph of $f(x)$ must have a corner at the point where $x = 0$.
14. If $y = f(x)$, then $\left.\frac{dy}{dx}\right|_{x=a} = f'(a)$.
15. If you zoom in (with your calculator) on the graph of $y = f(x)$ in a small interval around $x = 10$ and see a straight line, then the slope of that line equals the derivative $f'(10)$.
16. If $f''(x) > 0$ then $f'(x)$ is increasing.
17. The instantaneous acceleration of a moving particle at time t is the limit of difference quotients.
18. If $f(t)$ is the quantity in grams of a chemical produced after t minutes and $g(t)$ is the same quantity in kilograms, then $f'(t) = 1000g'(t)$.
19. If $f(t)$ is the quantity in kilograms of a chemical produced after t minutes and $g(t)$ is the quantity in kilograms produced after t seconds, then $f'(t) = 60g'(t)$.
20. A function which is monotonic on an interval is either increasing or decreasing on the interval.
21. The function $f(x) = x^3$ is monotonic on any interval.
22. The function $f(x) = x^2$ is monotonic on any interval.

Are the statements in Problems 23–27 true or false? If a statement is true, give an example illustrating it. If a statement is false, give a counterexample.

23. There is a function which is continuous on $[1, 5]$ but not differentiable at $x = 3$.
24. If a function is differentiable, then it is continuous.
25. If a function is continuous, then it is differentiable.
26. If a function is not continuous, then it is not differentiable.
27. If a function is not differentiable, then it is not continuous.
28. Which of the following would be a counterexample to the statement: "If f is differentiable at $x = a$ then f is continuous at $x = a$"?
    - **(a)** A function which is not differentiable at $x = a$ but is continuous at $x = a$.
    - **(b)** A function which is not continuous at $x = a$ but is differentiable at $x = a$.
    - **(c)** A function which is both continuous and differentiable at $x = a$.
    - **(d)** A function which is neither continuous nor differentiable at $x = a$.